

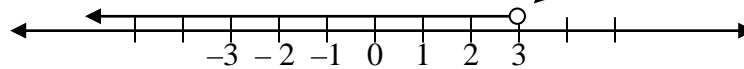
# LINEAR INEQUALITIES

When we use the equal sign in an equation we are stating that both sides of the equation are equal to each other. In an **inequality**, we are stating that both sides of the equation *are not* equal to each other. It can also be seen as an order relation; that is, it tells us which one of the two expressions is smaller, or larger, than the other one. A **linear inequality** is an equation in which the highest variable exponent is one.

<i>Examples:</i>	$x < 3$	→	less than, <
	$2x + 5 \leq x - 3$	→	less than or equal to, ≤
	$x > -2$	→	greater than, >
	$3x - 2 \geq x - 6$	→	greater than or equal to, ≥

The solution to an inequality is the value of the variable which makes the statement, or the inequality, true.

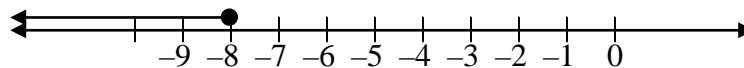
*Examples:*  $x < 3$  → This inequality is telling us that “x is less than 3”. Therefore, any number less than three is a possible solution. Remember that on the number line, any number to the left is less than a given number, and any number to the right of that given number is greater.



Notice how 3 is marked with an empty dot, this means that the answer does not include the number 3.

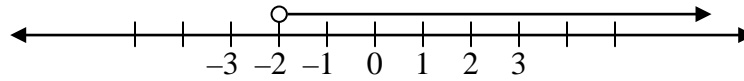
$2x + 5 \leq x - 3$  → This inequality is telling us that the equation  $2x + 5$  is less than or equal to the equation  $x - 3$ . Solve for  $x$  to find which number, or numbers, make this equation true.

$2x + 5 \leq x - 3$	→	Solve for $x$ by leaving the variables on one side
$2x + 5 - 5 \leq x - 3 - 5$		and the numbers on the other side, just as you
$2x \leq x - 8$		would solve a linear equation.
$2x - x \leq x - x - 8$		
$x \leq -8$		

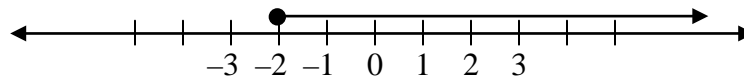


Notice how -8 is marked with a black dot, this means that the answer does include the number -8.

$x > -2 \rightarrow$  This inequality is telling us that “ $x$  is greater than  $-2$ ”.  
 Therefore,  
 any number greater than negative two is a possible solution.



$$\begin{aligned}
 3x - 2 &\geq x - 6 && \rightarrow \text{Solve for } x \text{ by leaving the variables on one side} \\
 3x - 2 + 2 &\geq x - 6 + 2 && \text{and the numbers on the other side.} \\
 3x &\geq x - 4 \\
 3x - x &\geq x - x - 4 \\
 2x &\geq -4 \\
 \frac{2x}{2} &\geq \frac{-4}{2} \\
 x &\geq -2
 \end{aligned}$$



As you can see, solving linear inequalities is very similar to solving linear equations. There is only one thing you need to keep in mind when multiplying or dividing a negative number, the direction or sense of the inequality is reversed. For example, if the sign is  $>$ , after multiplying or dividing by a negative number the sign changes to  $<$ .

*Example:*

$$\begin{aligned}
 \frac{1}{2}x + 4 &\geq x - 2 && \rightarrow \text{Solve for } x \text{ as in the previous examples.} \\
 \frac{1}{2}x + 4 - 4 &\geq x - 2 - 4 \\
 \frac{1}{2}x &\geq x - 6 \\
 \frac{1}{2}x - x &\geq x - x - 6 \\
 -\frac{1}{2}x &\geq -6 && \rightarrow \text{Multiply times the negative reciprocal to} \\
 &&& \text{leave the } x \text{ alone and get an answer.} \\
 -\frac{2}{1} \cdot -\frac{1}{2}x &\geq -6 \cdot -\frac{2}{1} && \text{However, since we are multiplying by a} \\
 &&& \text{negative the inequality sign must change} \\
 &&& \text{from } \geq \text{ to } \leq . \\
 x &\leq 12
 \end{aligned}$$

## EQUATIONS INVOLVING ABSOLUTE VALUE

The absolute value of a number is always the positive value of that number. Since the absolute value is the distance of a number from the origin, and since distances are always positive, the absolute value is always a positive value. The same definition applies to equations involving absolute value. Any equation inside the absolute value bars must always be equal to a positive number.

*Example:*  $|x - 4| = -8 \rightarrow$  This absolute value equation does not have a real answer.

Any equation inside the absolute value bars must always be equal to a positive number. In this case, for  $x - 4$  to be equal to  $-8$ ,  $x$  must be  $-4$ :  $-4 - 4 = -8$

However, since the equation  $x - 4$  is inside the absolute value bars it will always yield a positive answer  $|-8| = 8$  and never a negative answer.

When the equation involving absolute value is equal to a positive number, the value of the variable inside the absolute value bars can be negative or positive; therefore, the equation must be made equal to a positive and a negative answer. The answer will have two different possible values of  $x$ , and both will make the statement valid.

*Examples:*  $|-3x + 6| = 12 \rightarrow$  Make the equation equal to 12 and  $-12$  and solve for  $x$  in each case.

$$\begin{aligned} -3x + 6 &= 12 \\ -3x + 6 - 6 &= 12 - 6 \\ -3x &= 6 \\ \frac{-3x}{-3} &= \frac{6}{-3} \\ x &= -2 \end{aligned}$$

**and**

$$\begin{aligned} -3x + 6 &= -12 \\ -3x + 6 - 6 &= -12 - 6 \\ -3x &= -18 \\ \frac{-3x}{-3} &= \frac{-18}{-3} \\ x &= 6 \end{aligned}$$

The answer to  $|-3x + 6| = 12$  is  $x = -2$  and  $x = 6$ .

$$\begin{aligned} 6 + |5x - 6| &= 10 \\ |5x - 6| &= 4 \end{aligned}$$

$\rightarrow$  Leave the absolute value expression by itself on the left side of the equation. Then solve for  $x$  as in the previous example.

$$\begin{aligned} 5x - 6 &= 4 \\ 5x - 6 + 6 &= 4 + 6 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

**and**

$$\begin{aligned} 5x - 6 &= -4 \\ 5x - 6 + 6 &= -4 + 6 \\ 5x &= 2 \\ x &= \frac{2}{5} \end{aligned}$$

The answer is  $x = 2$  and  $x = \frac{2}{5}$ .

## INEQUALITIES INVOLVING ABSOLUTE VALUE

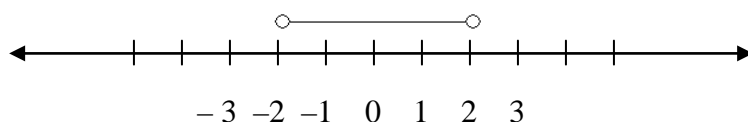
Inequalities involving absolute value are solved in a similar manner to equations involving absolute value. Since the variable inside the absolute value sign can be either positive or negative, both positive and negative values are used to solve for the variable. However, in the case of linear inequalities involving absolute value, the inequality is placed between the two values rather than in two different equations.

*Examples:*

$|x| < 2 \quad \rightarrow$  To solve this linear inequality use both positive and negative values of 2 and place the inequality in the middle without the absolute value sign. The inequality sign will be the same on both sides.

$-2 < x < 2 \quad \rightarrow$  This notation is read: *x is greater than -2 but less than*

2



$|-4x + 10| \geq 2 \quad \rightarrow$  Write both positive and negative values of 2 on each side of the inequality and leave it in the middle.

$-2 \geq -4x + 10 \geq 2 \quad \rightarrow$  Solve for  $x$  by moving each term to both sides of the inequality.

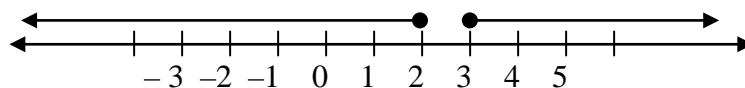
$$-2 - 10 \geq -4x + 10 - 10 \geq 2 - 10$$

$$-12 \geq -4x \geq -8$$

$$\frac{-12}{-4} \geq \frac{-4x}{-4} \geq \frac{-8}{-4}$$

$\rightarrow$  Remember that when dividing by a negative number the direction of the inequality sign changes.

$3 \leq x \leq 2 \quad \rightarrow$  This notation is read: *x is greater than 3 but less than 2.*



We explained previously that equations that contain an absolute value cannot be equal to a negative number since an absolute value will always give you a positive number. In the case of linear inequalities, if an inequality is greater than or equal to a negative number, then there are infinite solutions. If the inequality is less than or equal to a negative number, then there is no real solution to the inequality.

*Examples:*                     $|x + 2| \geq -2$      $\rightarrow$     The absolute value will give you a positive number which will always be greater than any negative number.

The inequality is true and has *infinite solutions*.

$|x + 2| \leq -2$      $\rightarrow$     The absolute value will give you a positive number and no positive number will ever be less than or equal to a negative number.

The inequality is false and has *no real solution*.

## LINEAR INEQUALITIES AND ABSOLUTE VALUE EQUATIONS – EXERCISES

**Solve for x:**

1.  $3x \leq 27$

2.  $-5x \leq 40$

3.  $-6x + 3 \geq 21$

4.  $-\frac{2}{3}x \geq 10$

5.  $3 - 2x \geq -8 + 3x$

6.  $3(2x + 1) < -8$

7.  $-(x + 1) - 4x > 4x - 3$

8.  $-3 \leq 4x + 2 \leq 10$

9.  $-3 \leq 2x + \frac{1}{3}x + 4 \leq 6$

10.  $|3x - 7| - 5 = 12$

11.  $\left|2 - \frac{x}{3}\right| = 4$

12.  $|x| \geq 4$

13.  $|2x - 3| \leq 6$

## LINEAR INEQUALITIES AND ABSOLUTE VALUE EQUATIONS – ANSWER TO EXERCISES

1.  $3x \leq 27$   
 $x \leq 9$

2.  $-5x \leq 40$   
 $x \geq -8$

3.  $-6x + 3 \geq 21$   
 $-6x \geq 21 - 3$   
 $-6x \geq 18$   
 $x \leq -3$

4.  $-\frac{2}{3}x \geq 10$   
 $-2x \geq 30$   
 $x \leq -15$

5.  $3 - 2x \geq -8 + 3x$   
 $-2x - 3x \geq -8 - 3$   
 $-5x \geq -11$   
 $x \leq \frac{11}{5}$

6.  $3(2x + 1) < -8$   
 $6x + 3 < -8$   
 $6x < -11$   
 $x < -\frac{11}{6}$

7.  $-(x + 1) - 4x > 4x - 3$   
 $-x - 1 - 4x > 4x - 3$   
 $-9x > -2$   
 $x < \frac{2}{9}$

8.  $-3 \leq 4x + 2 \leq 10$   
 $-3 - 2 \leq 4x \leq 10 - 2$   
 $-5 \leq 4x \leq 8$   
 $-\frac{5}{4} \leq x \leq 2$

9.  $-3 \leq 2x + \frac{1}{3}x + 4 \leq 6$   
 $-9 \leq 6x + x + 12 \leq 18$   
 $-9 - 12 \leq 7x \leq 18 - 12$   
 $-21 \leq 7x \leq 6$   
 $-3 \leq x \leq \frac{6}{7}$

10.  $|3x - 7| - 5 = 12$

$|3x - 7| = 12 + 5$

$|3x - 7| = 17$

$3x - 7 = 17$

$3x = 24$

$x = 8$

$3x - 7 = -17$

$3x = -10$

$x = -\frac{10}{3}$

**or**

11.  $\left|2 - \frac{x}{3}\right| = 4$

$2 - \frac{x}{3} = 4$

$-\frac{x}{3} = 2$

$x = -6$

$2 - \frac{x}{3} = -4$

$-\frac{x}{3} = -6$

$x = 18$

**or**

12.  $|x| \geq 4$   
 $-4 \geq x \geq 4$

13.  $|2x - 3| \leq 6$   
 $-6 \leq 2x - 3 \leq 6$   
 $-6 + 3 \leq 2x \leq 6 + 3$   
 $-3 \leq 2x \leq 9$   
 $-\frac{3}{2} \leq x \leq \frac{9}{2}$